The oriented chromatic number of some grids

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Abstract

We define some infinite subfamily of hexagonal grids with the oriented chromatic number 5. We present an algorithm for oriented colouring of some hexagonal planar oriented grids. The algorithm uses BFS spanning tree of a subgraph of the dual graph of the grid and a homomorphism to some tournament of order 6. In general the difference between the number of colours given by the algorithm and the oriented chromatic number is at most 1.

1. Introduction

We consider simple graphs and oriented digraphs. The simple graph $G = (V,E)$ is the pair of sets, where the second set consists of some two element subsets (edges) of the first set. The digraph $D = (V,A)$ is the pair of sets, where the second set consists of some pairs of elements (arcs) from the first set. We say that D is the oriented digraph if A does not contain opposite arcs (if A is antisymmetric relation on V). The oriented colouring and its relation to another type of colourings of graphs and digraphs is studied recently. Most of the results are published in [1-14]. Some important application of oriented colouring to study antisymmetric flows is presented in [10,11]. In this paper we study the oriented chromatic number for some planar and non planar grids. In particular, we give infinite subfamily of hexagonal grids with the oriented chromatic number 5. We extend a result of Fertin [4] to some family of square grids. Moreover, we present an algorithm for oriented colouring of hexagonal planar grids with some constrains. The basic notions and notations are defined below. The notions not defined here one can find in [3].

Homomorphism of the oriented digraph $D_1 = (V_1, A_1)$ to the oriented digraph $D_2 = (V_2, A_2)$ is defined as mapping φ from the vertex set of D_1 to the vertex set of D_2 satisfying the condition:

if (x, y) is an arc in D_1, then $(\varphi(x), \varphi(y))$ is an arc in D_2.

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Two oriented digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic* if there exists a bijective mapping φ from V_1 to V_2 satisfying the following condition:

$$\forall_{a,b \in V_1} \big((a,b) \in A_1\big) \Leftrightarrow \big((\varphi(a), \varphi(b)) \in A_2\big).$$

The proper vertex colouring of the graph G = (V,E) is a mapping $c : V \to N$ such that the adjacent vertices $u, v \in V$ have different colours, i.e., $\{u, v\} \in E \Rightarrow c(u) \neq c(v)$.

Given the positive integer k, if there exists a proper vertex colouring of G with k colours, then we say that G is *k-colourable*. The minimum number k, such that the graph G is k-colourable is *called the chromatic number of* G and denoted by χ(G).

The oriented colouring is a colouring c of the vertex set of the oriented digraph $\vec{G} = (V(\vec{G}), A(\vec{G}))$ satisfying the following conditions:

   – if $(u,v) \in A(\vec{G})$, then $c(u) \neq c(v)$,

   – for any pair of arcs $(u,v),(x,y) \in A(\vec{G})$, if $c(u)=c(y)$ then $c(x) \neq c(v)$.

The oriented chromatic number $\vec{\chi}(\vec{G})$ *of the oriented digraph* $\vec{G}$ is the minimum number of colours over all oriented colourings of $\vec{G}$.

Let O_G be the family of all possible orientations of the simple graph G. *The oriented chromatic number* $\vec{\chi}(G)$ *of the simple graph* G is defined as the maximum oriented chromatic number over all possible orientations of G, i.e.,

$$\vec{\chi}(G) = \max\left\{\vec{\chi}(\vec{G}) \mid \vec{G} \in O_G\right\}$$

Evidently, the oriented digraph D_1 can be coloured by k colours if and only if there exists a homomorphism φ from D_1 to an oriented digraph D_2 of order k. So the problem of finding the oriented chromatic number of an oriented digraph $D_1 = (V_1, A_1)$ is the problem of finding an oriented digraph $D_2 = (V_2, A_2)$ of minimum order, such that there exists a homomorphism $\varphi\colon D_1 \to D_2$. We say, that D_1 *is coloured by a homomorphism to* D_2. We say that D_1 is D_2 – *colourable*, if and only if there exists a homomorphism from D_1 to D_2. The vertices of D_2 are called the *colours*. The following digraphs are very useful for studying the oriented colourings of graphs. Two of them are presented in Figure 1.

   – T_5 is the digraph with the vertex set V = {0, 1,...,4} and the arc set
$$A = \left\{(x,(x+k)\bmod 5) \mid x \in V, \ k = 1,2\right\};$$

   – T_6 is the digraph with the vertex set V = {0, 1,...,5} and the arc set
$$A = \left\{(x,(x+1)\bmod 6): x \in V\right\} \cup \left\{(x,(x+k)\bmod 6): x \in V \wedge 2 \mid x, k = 2,3\right\} \cup$$
$$\left\{(x,(x+4)\bmod 6) \mid x \in V \wedge \neg 2 \mid x\right\};$$

- T_7 is the digraph with the vertex set $V = \{0, 1,...,6\}$ and the arc set
$$A = \left\{ \left(x, (x+k) \bmod 7 \right) \mid x \in V,\ k = 1, 2, 4 \right\};$$

- T_{11} is the digraph with the vertex set $V = \{0, 1,...,10\}$ and the arc set
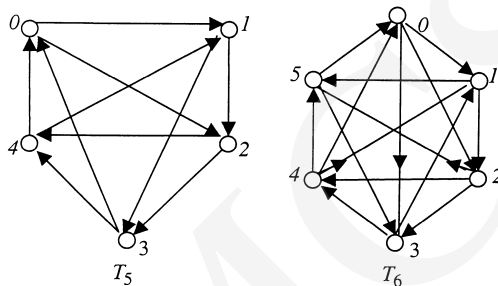$$A = \left\{ \left(x, (x+k) \bmod 11 \right) \mid x \in V,\ k = 1, 3, 4, 5, 9 \right\}.$$

Fig. 1. The orientation T_5 of the complete graph of order 5 and the orientation
T_6 of the complete graph of order 6

The average degree ad(H) of the graph H is defined as follows
$$ad(H) = \frac{2 \mid E(H) \mid}{\mid V(H) \mid}.$$

The maximum average degree mad(G) *of the graph* G is defined as maximum of average degrees ad(H) over all subgraphs H of the graph G, i.e. $mad(H) = \max\{ad(H) : H \subseteq G\}$.

The girth g(G) of the graph G is the order of the shortest cycle in G.

The following theorem presents some relation between the oriented chromatic number and the maximum average degree.

Theorem 1. (Borodin [2]).

1. For any graph G with $mad(G) < \dfrac{7}{3}$, $\vec{\chi}(G) \le 5$.

2. For any graph G with $mad(G) < \dfrac{11}{4}$ and $g(G) \ge 5$, $\vec{\chi}(G) \le 7$.

3. For any graph G with $mad(G) < 3$, $\vec{\chi}(G) \le 11$.

4. For any graph G with $mad(G) < \dfrac{10}{3}$, $\vec{\chi}(G) \le 19$.

2. The oriented colourings of square grids

In this section we give a short survey of known results for planar graphs, in particular for planar square grids. Moreover, we extend some of them for other square grids. The following theorems show some upper bounds for the oriented

chromatic number of planar graphs with respect to the maximum degree and the girth of the graph.

Theorem 2. (Raspaud and Sopena [9]). Any oriented planar digraph has an oriented colouring with at most $5 \cdot 2^4$ colours.

Theorem 3. (Kostochka [6]). *If G = (V,E) is a graph of maximum degree k, then*
$$\bar{\chi}(G) \leq 2 \cdot k^2 2^k.$$

Theorem 4. (Borodin [2]). *Let G be a simple planar graph with the girth g(G).*
   1) If g(G) > 13, then $\bar{\chi}(G) \leq 5$.
   2) If g(G) > 7, then $\bar{\chi}(G) \leq 7$.
   3) If g(G) > 5, then $\bar{\chi}(G) \leq 11$.
   4) If g(G) > 4, then $\bar{\chi}(G) \leq 19$

   The special subfamily of planar square grids are two-dimensional grids. *The two-dimensional grid G(m, n) is the cartesian product of two paths of orders m and n*, respectively, i.e., $G(m,n) = P_m \times P_n$ (see Figure 2 for some examples).

   The family of all G(m,n), where *m, n* are positive integers we denote by G_2. The oriented chromatic number for a family *F* of graphs is defined as follows:
$$\bar{\chi}(F) = \max\{\bar{\chi}(G) \mid G \in F\}.$$

   Some big two-dimensional grids can be optimally coloured by digraphs T_{11}, T_7 or T_6. The upper bound for the oriented chromatic number of G(m,n) in general case is given by the respective homomorphism to T_{11}. The nice property of T_{11} cited below is very useful to study the upper bound.
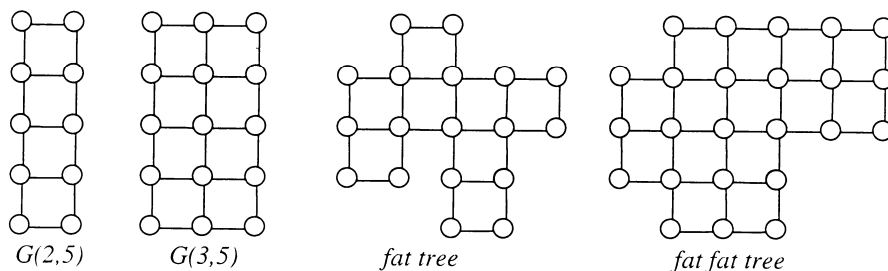

$G(2,5)$ $G(3,5)$ *fat tree*   *fat fat tree*

Fig. 2. Two two-dimensional grids *G(2,5)* and *G(3,5)*. The examples of fat tree and fat fat tree

Proposition 5. (Borodin, et al. [2]). *For any two vertices u, v of T_{11} there are at least two different paths of length 2 with an arbitrary orientation, joining the vertices u and v.*

Theorem 6. (Fertin, et al. [4]). *Let m, n be integers and let G_2 be the family of two- dimensional grids G(m, n), then $8 \leq \bar{\chi}(G_2) \leq 11$.*

The colouring of $G(2,n)$ by T_6 is discovered by Fertin, Raspaud and Roychowdhury [4]. For this narrow two-dimensional grid they obtained the result presented below.

Proposition 7. (Fertin, et al. [4]). For any $n>3$, $\vec{\chi}\big(G(2,n)\big)=6$ and $\vec{\chi}\big(G(2,2)\big)=4$, $\vec{\chi}\big(G(2,3)\big)=5$.

The following property of T_6 is very useful to study narrow two-dimensional grids. This property is very interesting for the oriented colouring of other grids studied in this paper, as well.

Proposition 8. *For any two vertices u and v of T_6 there exists a walk $P=(u,u',v',v)$ of length 3 with an arbitrary orientation of three arcs.*

Proof. Without loss of generality we can consider two cases for u, namely $u = 0$ and $u = 1$. □

Recently, Szepietowski and Targan [14] discovered the optimal oriented colouring of $G(3, n)$ and $G(4,n)$ by T_7. The digraph T_7 has some number of automorphisms, for example: $h(x) = (x + n) \bmod 7$, for any integer n; $h(x) = (2x) \bmod 7$, $h(x) = (4x) \bmod 7$, $h(x) = (7-x) \bmod 7$, where x is a vertex of T_7. The last automorphism reverses any arc of T_7, i.e., it maps any arc (u,v) into the arc (v,u), where u, v are any different vertices of T_7.

Proposition 9. (Szepietowski and Targan [14]). *There exists an automorphism of T_7 mapping any arc (u, v) of T_7 into the arc (0, 1).*

Proposition 10. (Szepietowski and Targan [14]). *Let G be an oriented two-dimensional grid and let (u,v) be an arc of G. Then the following theorems are equivalent:*

1) there exists a homomorphism h from G to T_7,

2) there exists a homomorphism h from G to T_7 so that h(u) = and h(v) = 1.

Proposition 11. (Fertin, et al. [4]). *For any orientation of the graph E_3 and any colouring of the vertices x, y, z with colours from the set {0,1,2,3,4,5,6} there exists a colouring of the vertices x', y', z' preserving the homomorphism to T_7, where the graph E_3 is presented in Figure 3.*

Proposition 12. (Szepietowski and Targan [14]). *For $n = 3,4,5,6,$, $\vec{\chi}\big(G(3,n)\big)=6$ and for any n>6, $\vec{\chi}\big(G(3,n)\big)=7$. For any G(4,n) there exist a homomorphism to T_7.*
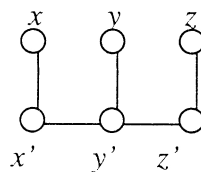

Fig. 3. The graph E_3

The above facts are very useful to study subgraphs of two-dimensional grids, called fat trees and fat fat trees and defined by Fertin, Raspaud and Roychowdhury in [4]. They glue grids G(2,n) into fat trees, and glue grids G(3,n) into fat fat trees with the restriction to the subgraphs of G(n,m) (see Figure 2 for the examples). The fat trees can be coloured in the same way as G(2,n). The fat fat trees can be coloured in the same way as G(3,n).

The oriented chromatic number of the family of fat trees and fat fat trees are cited below.

Proposition 13. (Fertin, et al. [4]). *Let* F_T *be the family of fat trees. Then* $\vec{\chi}(F_T) = 6$. *Let* FF_T *be the family of fat fat trees. Then* $\vec{\chi}(FF_T) = 7$.

We extend the above result to other square grids. Namely, let S be the family of graphs constructed from the cycle $C_4 = G(2,2)$ by successive edge gluing of a new copy of C_4. The family S contains the family of fat trees. Evidently each graph of the family is planar. Let FS be the family of graphs defined as follows. The smallest graph of the family FS is G(2,3). For $n>5$, the graph H of order $n+3$ belongs to FS if it can be constructed from a graph G of order n belonging to FS by gluing the vertices x,y,z of a new copy of E_3 to the consecutive vertices of a path P_3 in G. No other graph belongs to FS. The family FS contains all fat fat trees and some non planar square grids. The examples of graphs in the families are presented in Figures 2 and 4. The graphs presented in Figure 4 are neither fat tree nor fat fat tree.
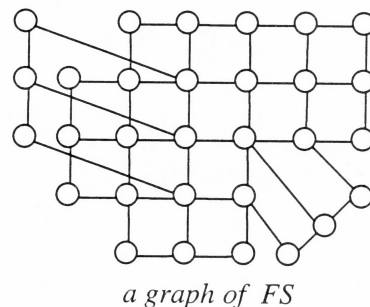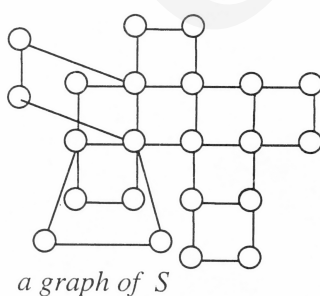

Fig. 4. The square grids of S and FS

Proposition 14. Let S and FS be the families defined above. Then $\vec{\chi}(S) = 6$ and $\vec{\chi}(FS) = 7$.

Proof. Immediately by Propositions 8 and 11. The graphs of the family S we colour by T_6, and the graphs of the family FS we colour by T_7.   □

3. Hexagonal grids

In this section we give some new results for oriented colouring of planar hexagonal grids. In particular we study the grids for which there exists a planar imbedding with at most one region with the boundary greater than 6 and such that any two hexagons have at most one common edge. The linear hexagonal grid $H_n = H_{1,n}$ is defined recursively as follows. $H_{1,1} = C_6$. For $n>1$, $H_{1,n}$ is constructed from $H_{1,n-1}$ by edge gluing of the new C_6 to the last right edge of $H_{1,n-1}$. If starting from C_6 we successively glue the new C_6 to an arbitrary edge of the hexagonal grid with n-1 hexagons then we obtain hexagonal tree with n hexagons. The 2-linear hexagonal grid $H_{2,n}$ is a particular case of fat hexagonal tree and is defined recursively as follows. $H_{2,1} = H_{1,2}$. For $n>1$, $H_{2,n}$ is constructed from $H_{2,n-1}$ by gluing of the new $H_{1,2}$ to the last right path P_4 of $H_{2,n-1}$. The examples of the linear hexagonal grid and the 2- linear hexagonal grid are presented in Figure 5. The last right edge is denoted by x, y in Figure 5(a), and the last right path P_4 is denoted by u, w, x, y in Figure 5(b).
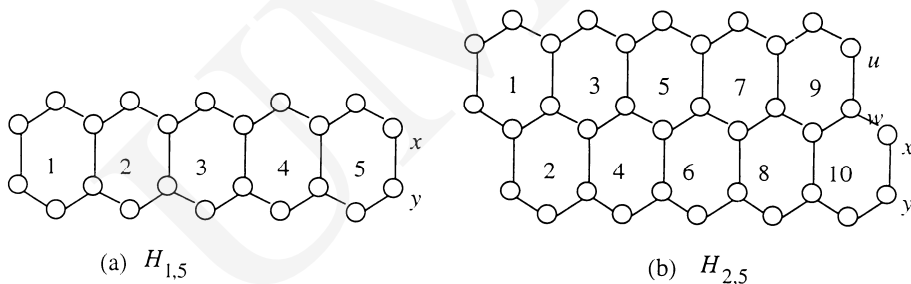

(a) $H_{1,5}$ (b) $H_{2,5}$

Fig. 5. The planar imbedding of $H_{1,5}$ and $H_{2,5}$. The numbers inside the hexagons are their labels. The optimal oriented colouring is constructed by extending the partial colouring by homomorphism to T_5 for successive hexagons according to increasing labels

First we prove the following lemma.

Lemma 15. *Let t be an integer, $4<t$. For any orientation of the path $P_t = (x_0, x_1, x_2, x_3, \dots, x_{t-1})$ joining the vertices x_0 and x_{t-1}, and for any colouring of x_0 and x_{t-1} by colours of the set $\{0,1,2,3,4\}$ there exists a colouring of $x_1, x_2, x_3, \dots, x_{t-2}$ preserving the homomorphism to T_5.*

Proof. (By induction on t). First we show that the theorem is true for $t = 5$. By symmetry of T_5 we can set the colour 0 for x_0, *i.e.* $c(x_0) = 0$ Moreover, by the symmetry, it is sufficient to consider the orientations of P_5 presented in Figure 6. The respective colouring of the inside vertices of P_5 by T_5 are listed in the table. Suppose that the theorem is true for each positive integer $4<n<t$. By taking the vertex x_{t-2} with the colour $c(x_{t-2}) = (c(x_{t-1})+1)\bmod 5$ and $c(x_{t-2}) = (c(x_{t-2})-1)\bmod 5$, we extend the result from P_{t-1} to P_t. $\qquad\square$

$c(x_4) =$				
0	*1*	*2*	*3*	*4*
123	230	230	241	123
241	123	234	134	240
243	120	131	121	132
121	132	243	120	120
213	140	140	212	213
101	202	213	140	101
104	210	210	142	143
413	340	401	402	402
404	404	410	342	303
434	434	430	312	312

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

Fig. 6. The orientations of the path P_5 and the oriented colourings by T_5.
We assume that $c(x_0) = 0$

We prove the following result.

Theorem 16. *Let n be a positive integer. For any n>3 we have* $\vec{\chi}(H_n) = 5$.

Proof. Let O_4 be the family of all orientations of the simple graph K_4. We show that the Figure 7 presents the oriented hexagonal grid $\vec{H}_4^0$ without any homomorphism to any digraph of the family O_4. Suppose that $\vec{\chi}(\vec{H}_4^0) \leq 4$. Without loss of the generality we set $c(q) = 0$, $c(h) = 1$, $c(p) = 2$. Then $c(r) \in \{0,3\}$. Suppose that $c(r) = 0$. Then $c(s) = 3$, $c(f) = c(k) = 2$ and $c(x) \in \{0,1,3\}$. Note that $c(x) \neq 0$, else $c(y) > 3$ and we get a contradiction. If $c(x) = 3$ then $c(y) = 0$ and $c(g) > 3$, a contradiction. If $c(x) = 1$ then $c(a) > 3$, a contradiction. Thus $c(r) = 3$. Hence $c(i) = 1$, $c(s) = 0$, $c(k) = 2$ and $c(j) = 1$. Thus $c(w) > 3$, a contradiction.
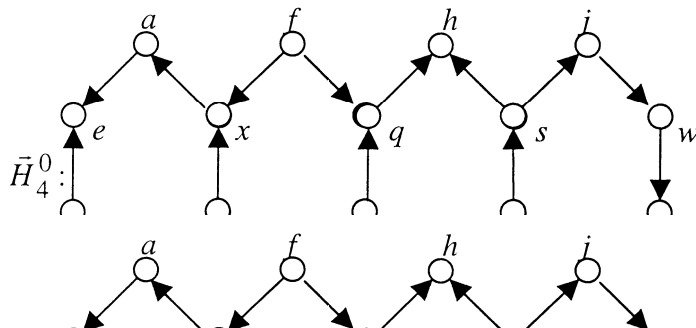

Fig. 7. The orientation $\vec{H}_4^0$ of the linear hexagonal grid $H_{1,4}$

So $\vec{\chi}(H_n) > 4$, for $n > 3$. By Lemma 15, note that for any orientation of $H_1 = H_{1,1} = C_6$ there exists a homomorphism to T_5. So $\vec{\chi}(H_1) \leq 5$. Thus, by induction on the number of hexagons and by Lemma 15 for $t = 6$, we get $\vec{\chi}(H_n) \leq 5$ for each positive integer n. The proof is done.   □

The hexagonal tree is obtained by edge gluing of linear hexagonal grids. *The fat hexagonal tree* is obtained by a P_4-gluing of 2-linear hexagonal grids. The examples are presented in Figure 8. More precisely, we construct the family of fat hexagonal trees recursively. We assume $H_{1,2}$ as the smallest fat hexagonal tree. The fat hexagonal tree H of order n, where $n > 6$ and $6|n$, we obtain from a fat hexagonal tree F of order n-6 by gluing the graph B_3 presented in Figure 9 to a path (u,w,x,y) of the graph F. The family contains non planar hexagonal grids. The oriented chromatic number for these families is presented below.

Theorem 17. *Let   HT be the family of all hexagonal trees. Then $\vec{\chi}(HT) = 5$.*

Proof. Immediately by Lemma 15 with $t = 6$.   □

Theorem 18. *Let FHT be the family of all fat hexagonal trees. Then $\vec{\chi}(FHT) = 5$.*

Proof. Let H be a graph of the family FHT and $\vec{H}$ be any orientation of H. We apply Lemma 15 with $t = 6$ and $t = 5$. First we colour the smallest oriented grid of the family by T_5. Then we extend the oriented colouring to the oriented colouring of $\vec{H}$ by the colouring of the sequence of respective orientations of subgraphs isomorphic to B_3 taking them accordingly to the recursive construction of H. First, we colour by T_5 the inside vertices of the path (u,a,b,c,x) and then we colour the inside vertices of the path (c,d,e,f,y).   □

The example for the respective order for the extending of the oriented colouring is presented in Figure 8.

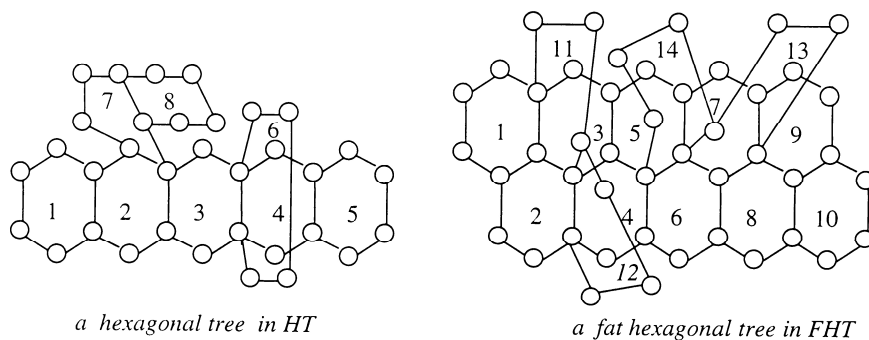a hexagonal tree in HT   a fat hexagonal tree in FHT

Fig. 8. The example of hexagonal tree and fat hexagonal tree. The numbers inside the hexagons are their labels. The optimal oriented colouring is constructed by extending the partial colouring by homomorphism to T_5 for successive hexagons according to the increasing labels
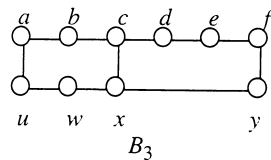
Fig. 9. The graph B_3. The vertices *u, w, x* and *y* are properly precoloured, i.e., the path
$P_4 = (u,w,x,y)$ is contained in *F*. The vertices *a, b, c, d, e* and *f* are coloured by T_5 as follows:
first we take the path (u,a,b,c,x) to extend the colouring to the vertices *a, b, c* and then
we take the path (c,d,e,f,y) to extend the colouring to the vertices *d, e, f*

To study other hexagonal grids we need the following lemma.

Lemma 19. *Let t be an integer, $3<t$. For any orientation of the path*
$P_t = (x_0, x_1, x_2, x_3, \ldots, x_{t-1})$ *joining the vertices x_0 and x_{t-1}, and for any colouring*
of x_0 and x_{t-1} by colours of the set {0,1,2,3,4,5} there exists a colouring of
$x_1, x_2, x_3, \ldots, x_{t-2}$ *preserving the homomorphism to T_6.*

Proof. (By induction on *t*). Immediately by Proposition 8 we get the result for
$t = 4$. Suppose that the theorem is true for each positive integer $3<n<t$. By taking
the vertex x_{t-2} with the colour $c(x_{t-2}) = (c(x_{t-1})+1) \bmod 6$ and $c(x_{t-2}) = (c(x_{t-1})-1) \bmod 6$ we extend the result from P_{t-1} to P_t. $\qquad\square$

Now we prove the following result.

Theorem 20. *Let H be a 2-connected planar hexagonal grid of maximum degree*
at most 3 with n, (n>3) hexagons such that any two hexagons have at most one
common edge and there exists planar imbedding of H with at most one non
hexagonal region. Then $5 \le \vec{\chi}(H) \le 6$.

Proof. Let *H* be a hexagonal planar grid with *n* hexagons satisfying the
assumption of the theorem. By Theorem 16 we have $5 \le \vec{\chi}(H)$. The upper bound
we get by the oriented colouring of *H* by T_6 according to the algorithm **OHPC**
presented below. The algorithm is based on Lemma 19. Let *R* be the graph with
the vertices representing the hexagons of *H*. Two vertices of *R* are adjacent if
and only if the respective hexagons have a common edge (unique one) in the
grid *H*. Let $\vec{H}$ be any orientation of *H*.

The algorithm OHPC (oriented colouring of oriented hexagonal planar grid
by T_6):

Imbed the grid *H* into the plane according to the assumption of the theorem.

Let *T* be a spanning BFS tree for *R*. {By BFS we mean the breadth first
search.}

Let $x_1, x_2, x_3, x_4, \ldots, x_n$ be the order of vertices of T obtained by BFS
algorithm.

FOR i =1(1)n **DO**

Take the noncoloured vertices of $\vec{H}$ from the boundary of the hexagon represented by x_i. The vertices create either an oriented path of order q, where $1<q<5$ (i.e., the vertices are the inside vertices of a path P_{q+2}), or an oriented cycle of order 6 (the last case is for the root of T). Denote the oriented subgraph by S.

For each vertex v of S find the respective homomorphism preserving colouring by T_6.

Note that the partial oriented colouring can be extended by Lemma 19.   □

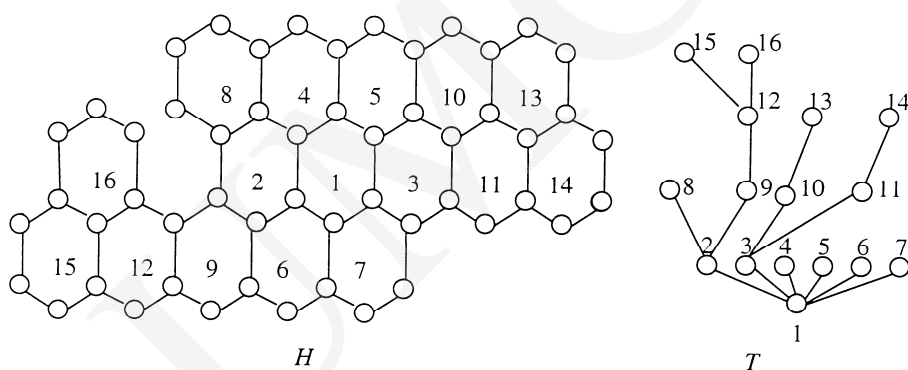The example of the grid satisfying the conditions of Theorem 20 is presented in Figure 10.

Fig. 10. The grid H with the respective BFS spanning tree T constructed in the increasing order of labels inside of hexagons

The algorithm OHPC presented in the proof of Theorem 20 can be applied to any oriented colouring of planar hexagonal grid H with the connectivity at most 1 such that each block of the grid satisfies the assumption of the theorem, as well. The block is the maximal 2-connected component of the graph. In the case we can use a spanning BFS forest F for R instead of a spanning BFS tree T and we colour the oriented grid according to some special order of trees of the forest. Namely, first we create a block representation tree B of the grid. The block tree has the blocks of H as vertices. Two vertices of B are adjacent iff the respective blocks have a common vertex. Then we number the vertices of B by BFS. We take the increasing order of the vertices of B as the order of blocks of H and we consider the blocks successively according to this order. For the first block we can create BFS tree with the root in the arbitrary hexagon. For each other block we create the respective BFS tree with the root representing the unique hexagon with precoloured vertex. The order of BFS trees is induced by the order of vertices in B. We apply the algorithm OHPC for each consecutive block.

If two hexagons of a planar grid *H* have at least two common edges or there exists no planar imbedding of H with at most one not hexagonal region, then the algorithm OHPC cannot produce any oriented colouring by T_6. The example of critical grid *F* is presented in Figure 11. Note that if after *9th* step the algorithm gives $c(u) = c(y)$, then the colouring cannot be extended to the vertex *x*. It follows by too small length of the path $P = (u,x,y)$. A critical orientation of the grid and the colouring by T_6 after *9th* step of the algorithm is presented in Figure 12.

Similarly, the grid obtained from *F* by deleting of the edges {*a,c*} and {*c,b*} is critical for the OHPC algorithm. In fact, there exists an infinite family of the critical graphs.
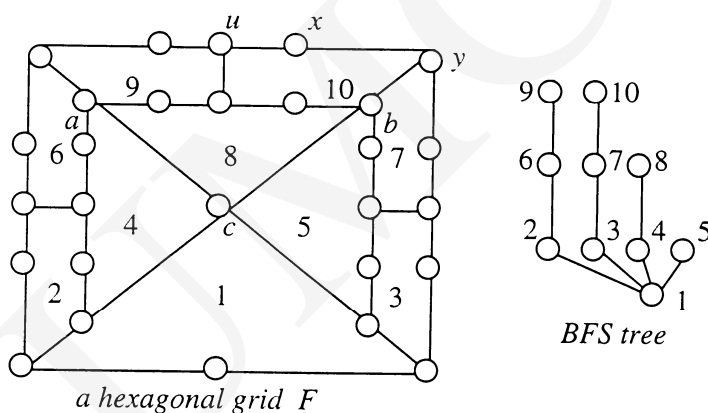

Fig. 11. The planar grid *F* without colouring by T_6 according to the algorithm OHPC. The respective BFS spanning tree *T* is constructed in the increasing order of labels inside hexagons
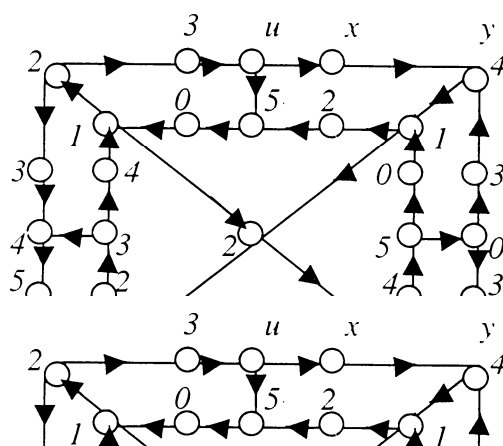

Fig. 12. A critical orientation $\vec{F}$ of the grid *F* and the colouring by T_6 after *9th* step of the algorithm. If $c(u) = 4$ then the orientation of the path $P = (u, x, y)$ leads to a non proper colouring of $\vec{F}$

Conlusions

Note that $mad(H_{1,n}) > 7/3$ and $mad(H_{2,n}) > 11/4$. So we cannot use Theorem 1 for the proof of Theorems 16-18. There is a question, what is the smallest grid satisfying the conditions of Theorem 20 with oriented chromatic number 6, (if such exists)? We conclude with the following open question for two-dimensional grids. What are the minimum positive integers *m* and *n* such that $\vec{\chi}(G(m,n)) = k$ for any *k* (5<k<12)?

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