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On bounded and involutive BRK-algebras

Abstract. In this article we introduce and analyze the concept of (involutive) bounded BRK-algebras. Additionally, we observe some of the substructures of (involutive) bounded BRK-algebras and find their mutual connections.

1. Introduction

In 1996, Imai and Iséki [8] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. These algebras have been extensively studied by many researchers since their introduction. In 1983, Hu and Li [9] introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2001, Neggers et al. ([13]) introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. This class of logical algebras is also known as RME (see, for example, [10], Definition 4.6(6)). R. Kumar Bandaru ([12]) introduced the concept of BRK-algebra, which is a generalization of BCK/BCI/BCH/Q-algebras. We know that every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a Q-algebra. Additionally, every Q-algebra is a BRK-algebra but converse needs not to be true ([12], Example 3.4). This class of logical algebras has been the focus of several researchers (see, for example [1, 3, 7, 15, 16]).

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The idea of a bounded BCK-algebra, that is, a BCK-algebra $\mathfrak{A} =: (A, *, 0)$ with the unit 1 which, in addition, satisfies the condition (F): $(\forall x \in A)$ $(x * 1 = 0)$ was the subject of research in [11] by K. Iséki. The boundedness of BE-algebra considered in [6] by Z. Çiloğlu and Y. Çeven and in [5] by R. Borzooei et al. The bounded GE-algebras were discussed in [4] by R. K. Bandaru et al. A. Walendziak discussed bounded/involutive WE-algebras in [18]. This author also analyzed the boundedness condition on QI-algebras in [14]. The above sources should justify our interest in observing some other bounded logical algebras.

In this article, we introduce and analyze the concepts of bounded and involutive BRK-algebras. It is shown that the direct product of an arbitrary family of bounded BRK-algebras is a bounded BRK-algebra again. Thus, for example, it has been stated that not every BRK-algebra can be extended to a bounded BRK-algebra. In fact, the bounded BRK-algebra is a special case of the bounded RML-algebra. (The concept of RML-algebra was introduced in [10], Definition 4.8.) Namely, a bounded BRK-algebra is a bounded RML-algebra which, additionally, satisfies the condition $(\forall x, y \in A)((x * y) * x = 0)$. The converse is not necessarily true.

Additionally, we consider some of the substructures of bounded BRK-algebras such as sub-algebras, (strong, weak) ideals, and filters in this class of logical algebras. Also, it is shown that the family of all strong ideals and the family of all weak ideals in a bounded BRK-algebra are mutually distinct.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, $\neg$, and so on, are labels for the logical functions of conjunction, disjunction, implication, negation, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use, as far as possible, their standard abbreviations that appear in a very well-known paper [10].

In this section, we define the notion of BRK-algebra.

Definition 2.1 ([12], Definition 3.1). A BRK-algebra is a nonempty set A with a constant 0 and a binary operation $*$ satisfying axioms:

(M) $\quad(\forall x \in A)(x * 0 = x)$,

(BRK) $(\forall x, y \in A)((x * y) * x = 0 * y)$.

We denote this system of axioms by **BRK** and the algebraic structure $\mathfrak{A} =: (A, *, 0)$ determined by it as BRK-algebra.

The following proposition contains some of the important properties of this class of logical algebras.

Proposition 2.1 ([12], Proposition 3.6, Proposition 3.7, Lemma 4.6). *If* $\mathfrak{A} =: (A, *, 0)$ *is a BRK-algebra, then the following formulas are valid:*

(Re) $(\forall x \in A)(x * x = 0)$,

(2) $(\forall x, y \in A)(x * y = 0 \implies 0 * x = 0 * y)$.

(3) $(\forall x, y \in A)(0 * (x * y) = (0 * x) * (0 * y))$ (0-left distributive).

(4) $(\forall x, y, z \in A)((x*y = x*z \implies 0*y = 0*z)$ (left semi-cancellative).

The concept of sub-algebras in BRK-algebras is determined by the standard way:

Definition 2.2 ([12], Definition 4.1). Let $\mathfrak{A} =: (A, *, 0)$ be a BRK-algebra. A non-empty subset S of A is called a sub-algebra of $\mathfrak{A}$ if

(S1) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x * y \in S)$.

It can be immediately concluded that it is valid

(S0) $0 \in S$.

Indeed, if S is a non-empty subset of A, then there exists at least some $x \in A$ such that $x \in S$. Then $0 = x * x \in S$ according to (S1) and (Re). We denote the family of all sub-algebras of the BRK-algebra $\mathfrak{A}$ by $\mathfrak{S}(A)$. It can be proved that the family $\mathfrak{S}(A)$ is a complete lattice.

Definition 2.3 ([12], Definition 4.20). Let $\mathfrak{A} =: (A, *, 0)$ be a BRK-algebra. A non-empty subset J of A is called an ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(J1) $(\forall x, y \in A)((x * y \in J \wedge y \in J) \implies x \in J)$.

We denote the family of all ideals of the BRK-algebra $\mathfrak{A}$ by $\mathfrak{J}(A)$. It is obvious that this family is not empty because $\{0\} \in \mathfrak{J}(A)$ and $A \in \mathfrak{J}(A)$. An ideal J in $\mathfrak{A}$ is said to be a non-trivial ideal in $\mathfrak{A}$ if $J \neq A$. Without major difficulties, it can be proved by the usual way that $\mathfrak{J}(A)$ is a complete lattice.

Example 2.1. Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows

$*$	0	a	b	c
0	0	a	0	a
a	a	0	a	0
b	b	a	0	a
c	c	b	c	0

It is easy to verify that $\mathfrak{A} =: (A, *, 0)$ is a BRK-algebra ([12], Example 3.4).

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$ and $S_4 = \{0, a, b\}$ are sub-algebras in $\mathfrak{A}$, while the subsets $\{0, c\}$, $\{0, a, c\}$ and $\{0, b, c\}$ are not sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, S_1, S_2, S_4, A\}$.

Subsets $J_0 = \{0\}$, $J_2 = \{0, b\}$ are ideals in $\mathfrak{A}$, while the subsets $\{0, a\}$, $\{0, c\}$, $\{0, a, b\}$, $\{0, a, c\}$ and $\{0, b, c\}$ are not ideals in $\mathfrak{A}$. So, $\mathfrak{J}(A) = \{J_0, J_2, A\}$. $\square$

The previous example shows that the families $\mathfrak{S}(A)$ and $\mathfrak{J}(A)$ are not coincident in BRK-algebras. Thus, the subset $S_1 = \{0, a\}$ is a sub-algebra in $\mathfrak{A}$ but it is not an ideal in $\mathfrak{A}$ because, for example, we have $b * a = a \in S_1$ but $b \notin S_1$.

Example 2.2. Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows

$*$	0	a	b	c
0	0	b	b	0
a	a	0	0	b
b	b	0	0	b
c	c	a	a	0

Then $\mathfrak{A} =: (A, *, 0)$ is a BRK-algebra ([12], Example 3.5).

Subsets $S_0 = \{0\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$ and $S_4 = \{0, a, b\}$ are sub-algebras in $\mathfrak{A}$, while the subsets $S_1 = \{0, a\}$, $S_5 = \{0, a, c\}$ and $S_6 = \{0, b, c\}$ are not sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, S_2, S_3, S_4, A\}$.

Subsets $J_0 = \{0\}$, $J_3 = \{0, c\}$ are ideals in $\mathfrak{A}$, while the subsets $\{0, a\}$, $\{0, b\}$, $\{0, a, b\}$, $\{0, a, c\}$ and $\{0, b, c\}$ are not ideals in $\mathfrak{A}$. Thus, for the sake of illustration, the subset $J_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a * b = 0 \in J_6$ but $a \notin J_6$. So, $\mathfrak{J}(A) = \{J_0, J_3, A\}$. $\square$

However, an ideal in the BRK-algebra need not be a sub-algebra, as the following example shows.

Example 2.3. Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	c	0	b
c	c	c	c	0

Then $\mathfrak{A} =: (A, *, 0)$ is a BRK-algebra. Subset $J_4 = \{0, a, b\}$ is an ideal in $\mathfrak{A}$, but it is not a sub-algebra in $\mathfrak{A}$ because, for example, we have $b * a = c \notin J_4$. $\square$

In the BRK-algebra $\mathfrak{A} = (A, *, 0)$ we introduce the relation $\preccurlyeq$ by the following way

$$(\forall x, y \in A)(x \preccurlyeq y \iff x * y = 0).$$

This relation is reflexive, due to the presence of (Re). This determination allows us to prove the following lemma.

Lemma 2.1. *Let J be an ideal of a BRK-algebra $\mathfrak{A} = (A, *, 0)$. Then*

(J2) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \implies x \in J)$.

Definition 2.4 ([3], Definition 3.2.7). Let $\mathfrak{A} =: (A, *, 0_A)$ and $\mathfrak{B} =: (B, \star, 0_B)$ be BRK-algebras. A mapping $f : A \longrightarrow B$ is said to be a homomorphism if

(f1) $(\forall x, y \in A)(f(x * y) = f(x) \star f(y))$.

It can be proved that

(f0) $f(0_A) = 0_B$.

Indeed, for arbitrary $x \in A$, we have $f(0_A) = f(x * x) = f(x) \star f(x) = 0_B$.

A homomorphism from the BRK-algebra $\mathfrak{A} = (A, *, 0_A)$ to the BRK-algebra $\mathfrak{B} = (B, \star, 0_B)$ is denoted by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$.

3. On bounded BRK-algebras

The determination of the concept of bounded BRK-algebras is given by the following definition.

Definition 3.1. A BRK-algebra $\mathfrak{A} =: (A, *, 0)$ is a bounded BRK-algebra if there is the element $1 \in A$ such that

(F) $(\forall x \in A)(x * 1 = 0)$.

If $\mathfrak{A}$ is a bounded BRK-algebra, the element 1, which satisfies the condition (F), we call the unit in $\mathfrak{A}$. In that case, we write $\mathfrak{A} =: (A, *, 0, 1)$.

The first thing to note about bounded BRK-algebras is given by the following lemma.

Lemma 3.1. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra. Then*

(L) $(\forall x \in A)(0 * x = 0)$.

Proof. Let $\mathfrak{A}$ be a bounded BRK-algebra. If we put $y = 1$ in (BRK), for arbitrary $x \in A$, we get $0 * x = (x * 1) * x = 0 * 1 = 0$ with respect to (BRK) and (F). □

The concept of RML-algebra was introduced in [10], Definition 4.8, as algebra $(A, *, 0)$ verifying the axioms (Re), (M), (L).

Conclusion 1. *Respecting the result of Lemma 3.1, the axiomatic system that determines bounded BRK-algebras, now has the form:*

(Re) $(\forall x \in A)(x * x = 0)$,

(M) $(\forall x \in A)(x * 0 = x)$,

(L) $(\forall x \in A)(0 * x = 0)$,

(F) $(\forall x \in A)(x * 1 = 0)$,

(BRK') $(\forall x, y \in A)((x * y) * x = 0)$.

So, it is a bounded RML-algebra which, in addition, satisfies the condition (BRK')*.*

As a consequence of the previous lemma, considering Examples 2.1 and 2.2, we have the second conclusion:

Conclusion 2. *In the general case, every BRK-algebra cannot be extended to a bounded BRK-algebra.*

Let us now do a little analysis in order to estimate the product $1 * y$ for an arbitrary $y \in A$ in a BRK-algebra $\mathfrak{A}$.

Let us put $x = 1$ in (BRK). Then, for arbitrary $y \in A$, we get $(1 * y) * 1 = 0 * y$. Hence, $0 = (1 * y) * 1 = 0$ with respect to (F) and (L). This means $1 * y \leqslant 1$. Let us consider the possible options:

(a) $1 * y = 1$,

(b) $1 * y = y$,

(c) $1 * y = z$,

where $z \in A$.

Example 3.1. Let $A = \{0, a, b, c, 1\}$ and let the operation in A be determined as follows

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	c	0	b	0
c	c	c	0	0	0
1	1	1	1	1	0

$*_2$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	c	0	b	0
c	c	c	0	0	0
1	1	a	b	c	0

$*_3$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	c	0	b	0
c	c	c	0	0	0
1	1	b	c	a	0

Then $\mathfrak{A} =: (A, *, 0, 1)$, $\mathfrak{B} =: (A, *_2, 0, 1)$ and $\mathfrak{C} =: (A, *_3, 0, 1)$ are bounded BRK-algebras. In all three previous cases, the ordering relation $\leqslant$ is given by:

$$\{(0,0), (0,a), (0,b), (0,c), (0,1), (a,a), (a,1), (b,b), (b,1), (c,b), (c,c), (c,1), (1,1)\}.$$

However, a bounded RML-algebra does not have to be a bounded BRK-algebra as the following example shows.

Example 3.2. Let $A = \{0, a, b, c, 1\}$ and let the operation in A be determined as follows

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	0	0	a	0
c	c	0	0	0	0
1	1	b	a	c	0

Then $\mathfrak{A} =: (A, *, 0, 1)$ is a bounded RML-algebra but it is not a BRK-algebra because, for example, we have $(b * c) * b = a * b = a \neq 0$. □

In what follows, we deal with the creation of the direct product bounded BRK-algebras. Let $\{(A_i, *_i, 0_i, 1_i) : i \in I\}$ be a family of bounded BRK-algebras. If on the set

$$\prod_{i \in I} A_i =: \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $(\prod_{i \in I} A_i, \odot, f_0, f_1)$, where f_0 and f_1 were chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i) \text{ and}$$

$$(\forall i \in I)(f_1(i) =: 1_i).$$

Before we start working with direct products of bounded BRK-algebras, we say that the operation, determined in this way, is well defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 3.1. *The direct product of any family of bounded BRK-algebras, determined as above, is a bounded BRK-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of bounded BRK-algebra.

Let $f, g \in \prod_{i \in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(Re) $(f \odot f)(i) = f(i) *_i f(i) = 0_i$,

(M) $(f \odot f_0)(i) = f(i) *_i f_0(i) = f(i) *_I 0 = f(i)$.

(L) We have $(f_0 \odot f)(i) = f_0 *_i f(i) = 0_i *_i f(i) = f(i)$ according to (L). Then $f_0 \odot f = f$.

(F) We have $(f \odot f_1)(i) = f(i) *_i f_1(i) = f(i) *_i 1_i = 0_i = f_0(i)$ by (F) in $(A_i, *_i, 0_i, 1_i)$. Hence, $f \odot f_1 = f_0$.

(BRK') $((f \odot g) \odot f)(i) = ((f(i) *_i g(i)) *_i f(i) = 0_i = f_0(i)$.
Hence $(f \odot g) \odot f = f_0$.

Therefore, the structure $(\prod_{i \in I} A_i, \odot, f_0, f_1)$ is a bounded BRK-algebra with the unit f_1 since it satisfies all its axioms. □

The preceding theorem is a generalization of Chaper 5 in [3].

4. Sub-algebras and ideals in bounded BRK-algebras

We define the concept of a sub-algebra in a bounded BRK-algebra a little more complexly than it was done in Definition 2.2.

Definition 4.1. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a BRK-algebra. A nonempty subset K in A is a sub-algebra in $\mathfrak{A}$ if, in addition to the conditions (S1), it also satisfies the condition

(K0) $1 \in K$.

We denote the family of all sub-algebras in the bounded BRK-algebra $\mathfrak{A}$ by $\mathfrak{K}(A)$.

Definition 4.2. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a BRK-algebra. A nonempty subset S in a BRK-algebra $\mathfrak{A}$ that satisfies the condition (S1) is called an incomplete sub-algebra in $\mathfrak{A}$. The family of all incomplete sub-algebras in a BRK-algebra $\mathfrak{A}$ is denoted by $\mathfrak{S}(A)$.

It is clear that every sub-algebra in a bounded BRK-algebra $\mathfrak{A}$ is an incomplete sub-algebra in $\mathfrak{A}$ and the converse need not hold. As already shown in Section 2, a nonempty subset of the bounded BRK-algebra that satisfies the condition (S1) also satisfies the condition (S0).

The substructure of ideals in a bounded BRK-algebra is introduced by the standard way (Definition 2.3).

It immediately comes to mind:

Lemma 4.1. *If J is a non-trivial ideal in a bounded BRK-algebra $\mathfrak{A}$, then $1 \notin J$.*

Proof. Let J be a nontrivial ideal in $\mathfrak{A}$. If we assume that $1 \in J$, then for arbitrary $x \in A$ we would conclude that $x * 1 = 0 \in J$ implies $x \in J$. Therefore, $A = J$ which is the opposite of $J \neq A$. Therefore, it must be $1 \notin J$. □

Example 4.1. (a) Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_5 = \{0, a, c\}$, $S_6 = \{0, b, c\}$, and $S_7 = \{0, a, b, c\}$ are incomplete sub-algebras in $\mathfrak{A}$.

Subsets $K_0 = \{0, 1\}$, $K_1 = \{0, 1, a\}$, $K_2 = \{0, 1, b\}$, $K_3 = \{0, 1, a, c\}$, $K_5 = \{0, 1, a, c\}$, $K_6 = \{0, 1, b, c\}$ and $K_7 = \{0, a, b, c, 1\}$ are sub-algebras in $\mathfrak{A}$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_3 = \{0, c\}$ and $J_6 = \{0, b, c\}$ are ideals in $\mathfrak{A}$.

(b) Let $\mathfrak{B} =: (A, *_2, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $K_0 = \{0, 1\}$, $K_1 = \{0, 1, a\}$, $K_2 = \{0, 1, b\}$, $K_3 = \{0, 1, c\}$, $K_5 = \{0, 1, a, c\}$ $K_6 = \{0, 1, b, c\}$ and $K_7 = \{0, a, b, c, 1\}$ are sub-algebras in $\mathfrak{B}$.

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_5 = \{0, a, c\}$, $S_6 = \{0, b, c\}$, and $S_7 = \{0, a, b, c\}$ are incomplete sub-algebras in $\mathfrak{B}$.

The only non-trivial ideal in this bounded BRK-algebra is the incomplete sub-algebra $J_0 = S_0$.

(c) Let $\mathfrak{C} =: (A, *_3, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $K_0 = \{0, 1\}$ and $K_7 = \{0, a, b, c, 1\}$ are sub-algebras in $\mathfrak{C}$.

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_5 = \{0, a, c\}$, $S_6 = \{0, b, c\}$ and $S_7 = \{0, a, b, c\}$ are incomplete sub-algebras in $\mathfrak{C}$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$ and $J_3 = \{0, c\}$ are ideals in $\mathfrak{C}$. □

Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra. For arbitrary $y \in A$, put $y^- =: 1 * y$.

Proposition 4.1. *For every sub-algebra K in $\mathfrak{A}$ we have*

(5) $(\forall x \in A)(x \in K \implies x^- \in K)$.

Proof. Obvious. □

Proposition 4.2. *Let J be a non-trivial ideal in a bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then*

(6) $(\forall x \in A)(x \in J \implies x^- \notin J)$,

(7) $(\forall x \in A)(x^- \in J \implies x \notin J)$,

(8) $(\forall x \in A)(x * x^- \in J \implies x^- \notin J)$,

(9) $(\forall x \in A)(x^- * x \in J \implies x \notin J)$,

(10) $(\forall x, y \in A)(x \in J \implies x * y \in J)$.

Proof. (6): Let $x \in A$ be such that $x \in J$. Assume that $1 * x = x^- \in J$. Then we would have $1 \in J$ which is impossible, since J is a non-trivial ideal in $\mathfrak{A}$. So it must be $x^- \notin J$.

(7): Assume that $x^- \in J$ and $x \in J$. Then there would be $1 \in J$ which is not possible, since J is not a trivial ideal in $\mathfrak{A}$. So it must be $x \notin J$.

(8): Let $x * x^- \in J$ and $x^- \in J$ for some $x \in A$. Then $x \in J$ by (J1). From this, according to (6), we get $x^- \notin J$. We got a contradiction. So it must be $x^- \notin J$.

(9): Let $x^- * x \in J$ and $x \in J$ for some $x \in A$. Then, we have $x^- \in J$ by (J1), whence, by (7), it follows, $x \notin J$. The resulting contradiction invalidates the assumption $x \in J$. Therefore, $x \notin J$.

(10): Let J be an ideal in a bounded BRK-algebra $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$. Then $x * y \in J$ by (J2), since $x * y \leqslant x$ in accordance with (BRK'). □

Corollary 4.1. *Every ideal in a bounded BRK-algebra $\mathfrak{A}$ is an incomplete sub-algebra in $\mathfrak{A}$. This means $\mathfrak{J}(A) \subseteq \mathfrak{S}(A)$.*

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then $x * y \in J$ and $y * x \in J$ by (10). Hence, J is an incomplete sub-algebra in $\mathfrak{A}$. □

The converse of Corollary 4.1 may not be valid, as shown in Example 4.1(c). For example, the incomplete sub-algebra S_2 is not an ideal in $\mathfrak{A}$.

Theorem 4.1. *The family $\mathfrak{S}(A)/\mathfrak{J}(A)/\mathfrak{K}(A)$ forms a complete lattice.*

Proof. We will give a proof for the family $\mathfrak{S}(A)$. The proof for the family $\mathfrak{J}(A)/\mathfrak{K}(A)$ can be carried out analogously.

Let $\{S_i : i \in I\}$ be a family of incomplete sub-algebras in $\mathfrak{A}$. It is clear that $0 \in \cap_{i\in I} S_i$ holds due to the presence of (S0) for every $i \in I$. Let $x, y \in A$ be such that $x \in \cap_{i\in I} S_i$ and $y \in \cap_{i\in I} S_i$. This means $x \in S_i$ and $y \in S_i$ for every $i \in I$. Then $x * y \in S_i$ for each $i \in I$ according to (S1). Thus $x * y \in \cap_{\in I} S_i$.

If we denote by $\mathcal{Z}$ the family of all incomplete sub-algebras of the algebra $\mathfrak{A}$ that contain $\cup_{i\in I} S_i$, then $\cap\mathcal{Z}$ is an incomplete sub-algebra of the algebra $\mathfrak{A}$ that contains $\cup_{i\in I} S_i$, according to the first part of this proof.

If we put $\sqcap_{i\in I} S_i = \cap_{i\in I} S_i$ and $\sqcup_{i\in I} S_i = \cap\mathcal{Z}$, then $(\mathfrak{S}(A), \sqcap, \sqcup)$ is a complete lattice. □

5. Filters in bounded BRK-algebras

5.1. Let us introduce the concept of filters in a bounded BRK-algebra by a usual manner: Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra. A nonempty subset F of A is called a filter in $\mathfrak{A}$ if

(F1) $(\forall x, y \in A)((x \in F \wedge x * y \in F) \Longrightarrow y \in F)$.

One of the important properties of the filter in the bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$, introduced in this way, gives the following proposition:

Proposition 5.1. *Let F be a filter in a bounded BRK-algebra $\mathfrak{A} =: (A,*,0,1)$. Then $K_0 =: \{0, 1\} \subseteq F$.*

Proof. Since F is not empty, there exists at least some $x \in A$ such that $x \in F$. Then, from $x \in F$ and $x * 0 = x \in F$ we get $0 \in F$ according to (F1). On the other hand, from $x \in F$ and $x * 1 = 0$ we get $1 \in F$ by (F), (F1) and the first part of this proof. □

As an important consequence of the previous proposition we have: For any $x \in A$ and for any filter F in $\mathfrak{A}$, introduced as above, we have that from $0 * x = 0 \in F$ and $0 \in F$ it follows that $x \in F$ by the previous proposition and (F1). This gives $A = F$. Thus, the only filter in the bounded BRK-algebra $\mathfrak{A}$, determined in the described way, is the set A itself.

5.2. The second type of filter is taken from [2], Definition 2.9, and applied to the BRK-algebra class:

Definition 5.1. ([2], Definition 2.9) A filter of a bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$ is a non-empty subset F of A such that:

(Fc) $(\forall x, y \in A)((x \in F \wedge y \in F) \implies (y*(y*x) \in F \wedge x*(x*y) \in F))$,

(F2) $(\forall x, y \in A)((x \in F \wedge x*y = 0) \implies y \in F)$.

By $\mathfrak{F}_1(A)$ we denote the family of all filters in the BRK-algebra $\mathfrak{A}$.

Proposition 5.2. *Let F be a filter in a bounded BRK-algebra $\mathfrak{A} =: (A,*,0,1)$. Then*

(F0) $1 \in F$.

Proof. Since F is not empty, there exists at least some $x \in A$ such that $x \in F$. Then from $x \in F$ and $x * 1 = 0$ it follows that $1 \in F$ according to (F2). □

Proposition 5.3. *Let F be a non-trivial filter in a bounded BRK-algebra $\mathfrak{A} =: (A,*,0,1)$. Then*

(11) $0 \notin F$.

Proof. Assume that $0 \in F$. Then, for an arbitrary $x \in A$, $0 \in F$ and $0*x = 0$ would give $x \in F$ with respect to (L) and (F2). This means $A = F$ which is impossible if F is not a trivial filter in $\mathfrak{A}$. Thus, for a non-trivial filter F in $\mathfrak{A}$, it must be $0 \notin F$. □

Example 5.1. (a) Let $\mathfrak{A} =: (A,*,0,1)$ be a bounded BRK-algebra as in Example 3.1. This BRK-algebra has no non-trivial filters.

(b) Let $\mathfrak{B} =: (A,*_2,0,1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $F_0 =: \{1\}$, $F_1 =: \{1,a\}$ and $F_2 =: \{1,b\}$ are filters in $\mathfrak{A}$. Subset $F_3 =: \{1,c\}$ is not a filter in $\mathfrak{A}$ because, for example, we have $c \in F_3$ and $c *_2 b = 0$ but $b \notin F_3$. Subset $F_4 =: \{1,a,b\}$ is not a filter in $\mathfrak{A}$ because, for example, we have $a*(a*b) = a*a = 0 \notin F_4$. Also, subset $F_5 =: \{1,a,c\}$ is not a filter in $\mathfrak{A}$ because, for example, we have $c \in F_5$ and $c *_2 b = 0$ but $b \notin F_5$. Also, neither of the subsets $F_6 =: \{1,b,c\}$ and $F_7 =: \{1,a,b,c\}$ is a filter in $\mathfrak{A}$.

(c) Let $\mathfrak{C} =: (A,*_3,0,1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $F_0 =: \{1\}$ and $F_7 =: \{1,a,b,c\}$ are filters in $\mathfrak{A}$ and there are no other non-trivial filters besides these. □

Remark 5.1. Requirement (Fc) in Definition 5.1 can be written in the form

$$(\forall x, y \in A)((x \in F \wedge y \in F) \implies x*(x*y) \in F).$$

For the proof it is enough to just substitute x instead of y and y instead of x. (The author is grateful to the reviewer for this remark.)

Conclusion 3. *The families $\mathfrak{S}(A)$ and $\mathfrak{F}_1(A)$ differ from each other as shown in Examples 4.1 and 5.1.*

So, for example, the incomplete sub-algebra $K_7 = \{0,1,b,c\}$ in the BRK-algebra $\mathfrak{B}$ is not a filter in $\mathfrak{B}$. On the other hand, the filter $F_2 = \{1,b\}$ in $\mathfrak{B}$ is not a (incomplete) sub-algebra in $\mathfrak{B}$.

Also, the following conclusion can be stated.

Conclusion 4. *The families* $\mathfrak{K}(A)$ *and* $\mathfrak{F}_1(A)$ *differ from each other.*

The following proposition describes a specificity of filters in BRK-algebras.

Proposition 5.4. *Let* F *be a filter in a bounded BRK-algebra* $\mathfrak{A} =: (A,*,0,1)$. *Then*

(12) $(\forall x \in A)(x \in F \implies (x^-)^- \in F)$.

Proof. Let F be a filter in a BRK-algebra A and let $x \in A$ be such that $x \in F$. Then $1 * (1 * x) = (x^-)^- \in F$ by (Fc). □

Further on, we have:

Theorem 5.1. *The family* $\mathfrak{F}_1(A)$ *forms a complete lattice.*

Proof. Let $\{F_i : i \in I\}$ be a family of filters in $\mathfrak{A}$.

It is clear that $1 \in \cap_{i\in I} F_i$ holds due to the presence of (F0) for every $i \in I$.

(a) Let $x, y \in A$ be such that $x \in \cap_{i\in I} F_i$ and $y \in \cap_{i\in I} F_i$. This means $x \in F_i$ and $y \in F_i$ for every $i \in I$. Then $y*(y*x) \in F_i$ and $x*(x*y) \in F_i$ for each $i \in I$ according to (Fc). Thus $y*(y*x) \in \cap_{\in I} F_i$ and $x*(x*y) \in \cap_{\in I} F_i$.

Let $x, y \in A$ be such that $x \in \cap_{i\in I} F_i$ and $x * y = 0$. This means $x \in F_i$ and $x * y = 0$ for any $i \in I$. Thus $y \in F_i$ by (F2) for each $i \in I$, Hence, $y \in \cap_{i\in I} F_i$.

This shows that $\cap_{i\in I} F_i$ is a filter in $\mathfrak{A}$.

(b) If we denote by $\mathcal{Z}$ the family of all filters in the algebra $\mathfrak{A}$ that contain $\cup_{i\in I} F_i$, then $\cap\mathcal{Z}$ is a filter in the algebra $\mathfrak{A}$ that contains $\cup_{i\in I} F_i$, according to the first part of this proof.

(c) If we put $\sqcap_{i\in I} F_i = \cap_{i\in I} F_i$ and $\sqcup_{i\in I} F_i = \cap\mathcal{Z}$, then $(\mathfrak{F}_1(A), \sqcap, \sqcup)$ is a complete lattice. □

Theorem 5.2. *Let* $\{(A_i, *_i, 0_i, 1_i) : i \in I\}$ *be a family of bounded BRK-algebras,* K *be a subset of* I *and let* F_i *be a filter in* $(A_i, *_i, 0_i, 1_i)$ *for each* $i \in K$. *Then* $\prod_{i\in I} T_i$, *where* $T_i = F_i$ *for* $i \in K$ *and* $T_i = A_i$ *for* $i \in I\backslash K$, *is a filter in the BRK-algebra* $\prod_{i\in I} \mathfrak{A}_i =: (\prod_{i\in I} A_i, \odot, f_0, f_1)$.

Proof. If $K = \emptyset$, then $\prod_{i\in I} T_i = \prod_{i\in I} A_i$, so $\prod_{i\in I} T_i$ is certainly a filter in $\prod_{i\in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y \in \prod_{i\in I} A_i$ be such that $x \in \prod_{i\in I} T_i$ and $y \in \prod_{i\in I} T_i$. This means $x(i) \in F_i$ and $y(i) \in F_i$ for each $i \in K$. Then

$$(x \odot (x \odot y))(i) = x(i) *_i (x(i) *_i y(i)) \in F_i \text{ and}$$

$$(y \odot (y \odot x))(i) = y(i) *_i (y(i) *_i x(i)) \in F_i,$$

since F_i is a filter in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \odot (x \odot y) \in \prod_{i\in I} T_i$ and $y \odot (y \odot x) \in \prod_{i\in I} T_i$.

Let $x, y \in A$ be such that $x \in \prod_{i\in I} T_i$ and $x \odot y = f_0$. This means $x(i) \in F_i$ and $(x \odot y)(i) = x(i) *_i y(i) = f_0(i) = 0_i$ for each $i \in K$. Then $y(i) \in F_i$ for each $i \in K$ according (F2). Hence, $y \in \prod_{i\in I} T_i$.

As shown, $\prod_{i\in I} T_i$ is a filter in $\prod_{i\in I} \mathfrak{A}_i$. □

Example 5.2. Let $\mathfrak{B} =: (A, *, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. Then $\mathfrak{B} \times \mathfrak{B} =: (A \times A, \odot, (0, 0), (1, 1))$ be a bounded BRK-algebra by Theorem 3.1, where the operation $\odot$ on $A \times A$ is defined by

$$(\forall x, y, u, v \in A)((x, u) \odot (y, v) =: (x * y, u * v)).$$

Subset $F_2 =: \{1, b\}$ is a filter in $\mathfrak{B}$ as shown in Example 5.1(b). Therefore, by the previous theorem, the subsets $F_2 \times A$, $F_2 \times F_2$ and $A \times F_2$ are filters in $\mathfrak{B} \times \mathfrak{B}$. □

6. Congruences on bounded BRK-algebras

6.1. Congruence and kernel of congruence. In this section, we introduce the notion of translative ideals in a bounded BRK-algebra.

Definition 6.1 ([17], Definition 3.1). An ideal J of a bounded BRK algebra $\mathfrak{A} =: (A, *, 0, 1)$ is called a translative ideal in $\mathfrak{A}$ if it satisfies the condition

(tJ) $(\forall, y, z \in A)((x * y \in J \wedge y * x \in I)$
$$\Longrightarrow ((x * z) * (y * z) \in J \wedge (z * x) * (z * y) \in J).$$

Moreover, in this section we study congruence relation on a bounded BRK-algebra. Also, we observe some new substructures of bounded BRK-algebras associated with congruences.

Definition 6.2 ([17], Definition 4.1). Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra. An equivalence relation q on the set A is called a left congruence on $\mathfrak{A}$ if

$$(\forall x, y, z \in A)((x, y) \in q \Longrightarrow (z * x, z * y) \in q).$$

A right congruence on a bounded BRK-algebra $\mathfrak{A}$ can be defined analogously. For the equivalence of q on A, we say that it is a congruence on $\mathfrak{A}$ if it is both a left and a right congruence on $\mathfrak{A}$.

Following the procedure described in [17], Section 4, now we construct a congruence relation on a bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$ via a translative ideal J in $\mathfrak{A}$. Let J be a translative ideal in a bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$. According to [17], Theorem 4.4, the relation q on A, defined by

$$(\forall x, y \in A)((x, y) \in q_J \iff (x * y \in J \wedge y * x \in J)),$$

is a congruence on $\mathfrak{A}$. Given the congruence relation q_J on the BRK-algebra $\mathfrak{A}$, we use the notation $[x]$ for the equivalence class, determined by x i.e.

$[x] =: \{y \in A : (y, x) \in q_J\}$, and $A/q_J =: \{[x] : x \in A\}$. We determine the operation $\star$ on A/q_J as follows

$$(\forall x, y \in A)([x] \star [y] =: [x * y]).$$

Now we can design the structure $\mathfrak{A}/q_J =: (A/q, \star, [0], [1])$. According to [17], Theorem 4.2, $\mathfrak{A}/q_J$ is a BRK-algebra. Let us show that $\mathfrak{A}/q_J$ is a bounded BRK-algebra with the unit [1]. Indeed, for arbitrary $x \in A$, we have $[x] \star [1] = [x * 1] = [0]$.

We can summarize the above.

Theorem 6.1. *Let J be a translative ideal in a bounded BRK-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then the structure $\mathfrak{A}/q_J =: (A/q_J, \star, [0], [1])$ is a bounded BRK-algebra with the unit* [1].

Let us now consider what insights we can deduce about the kernel of (left, right) congruence on a bounded BRK-algebra $\mathfrak{A}$.

Theorem 6.2. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra and $q \subseteq A \times A$ be a relation on A.*

(a) *If q is a left congruence on $\mathfrak{A}$, then the kernel $[0]$ of q is an ideal in $\mathfrak{A}$.*

(b) *If q is a right congruence on $\mathfrak{A}$, then the kernel $[0]$ of q is an incomplete sub-algebra in $\mathfrak{A}$.*

(c) *If q is a congruence on $\mathfrak{A}$, then the following condition holds*

$$(\forall x, y, z \in A)(((x * y) * z \in [0] \wedge y \in [0]) \implies x * z \in [0]).$$

Proof. (a) Let q be a left congruence on $\mathfrak{A}$. It is clear that $0 \in [0]$ holds due to the reflexivity of the relation q. Let $x, y \in A$ be such that $x * y \in [0]$ and $y \in [0]$. This means $(x * y, 0) \in q$ and $(y, 0) \in q$. Then $(x * y, 0) \in q$ and $(x * y, x * 0) = (x * y, x) \in q$ since q is a left congruence on $\mathfrak{A}$ and with respect to (M). Thus $(0, x) \in q$ in accordance with the transitivity of the relation q. Hence, $x \in [0]$. So, $[0]$ is an ideal in $\mathfrak{A}$.

(b) Let q be a right congruence on $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in [0]$ and $y \in [0]$. This means $(x, 0) \in q$ and $(y, 0) \in q$. Then $(x * y, 0 * y) = (x * y, 0) \in q$ and $(y * x, 0 * x) = (y * x, 0) \in q$ with respect to (L). Thus $x * y \in [0]$ and $y * x \in [0]$. Hence, $[0]$ is an incomplete sub-algebra in $\mathfrak{A}$.

(c) Let q be a congruence on $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x*y) \star z \in [0]$ and $y \in [0]$. This means $((x * y) * z, 0) \in q$ and $(y, 0) \in q$. Then $(x * y, x * 0) = (x * y, x) \in q$ with respect to (M) and since q is a right congruence on $\mathfrak{A}$. Hence $((x * y) * z, x * z) \in q$, since q is a right congruence on $\mathfrak{A}$. Thus $(x * z, 0) \in q$ in accordance with the transitivity of the relation q. So, $x * z \in [0]$. □

Remark 6.1. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra and $q \subseteq A \times A$ be a congruence on $\mathfrak{A}$. Let $x, y \in A$ be such that $x \in [0]$ and $y \in [0]$. This means that $(x, 0) \in q$ and $(y, 0) \in q$. Then $(y * x, y * 0) = (y * x, y) \in q$ and $(x * y, x * 0) = (x * y, x) \in q$ according to (M) and since q is a left congruence on $\mathfrak{A}$. Thus we have $(y * (y * x), y * y) = (y * (y * x), 0) \in q$ and $(x * (x * y), x * x) = (x * (x * y), 0) \in q$ with respect to (Re) and since q is a left congruence in $\mathfrak{A}$. Therefore, $y * (y * x) \in [0]$ and $x * (x * y) \in [0]$. This means that the kernel $[0]$ satisfies the condition (Fc).

Since $[0]$ is an ideal in $\mathfrak{A}$, we have $1 \notin [0]$ according to Lemma 4.1. Therefore, $[0]$ is not a filter in $\mathfrak{A}$.

6.2. Strong ideal. The condition, which is satisfied by the kernel $[0]$, together with the condition $0 \in [0]$, which is certainly valid, determine a strong ideal in a bounded BRK-algebra $\mathfrak{A}$.

Definition 6.3. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra. A nonempty subset J in A is called a strong ideal in $\mathfrak{A}$ if it satisfies the conditions (J0) and

(Js) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \Longrightarrow x * z \in J)$.

We denote the family of all strong ideals in a bounded BRK-algebra $\mathfrak{A}$ by $\mathfrak{J}_s(A)$.

Proposition 6.1. *Every strong ideal in a bounded BRK-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$. This means that $\mathfrak{J}_s(A) \subseteq \mathfrak{J}(A)$.*

Proof. If we put $z = 0$ in (Js), we get (J1). □

Proposition 6.2. *The subset $J_0 = \{0\}$ is a strong ideal in every bounded BRK-algebra $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $(x * y) * z = 0$ and $y = 0$. Then $0 = (x * 0) * z = x * z$. So, $\{0\}$ is a strong ideal in $\mathfrak{A}$. □

An ideal in a bounded BRK-algebra $\mathfrak{A}$ need not be a strong ideal in $\mathfrak{A}$ as the following example shows.

Example 6.1. (a) Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. The ideal $J_1 = \{0, a\}$ in $\mathfrak{A}$ is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(b*a)*c = c*c = 0 \in J_1$ and $a \in J_1$ but $b*c = b \notin J_1$. The ideals $J_3 =: \{0, c\}$ and $J_6 =: \{0, b, c\}$ in $\mathfrak{A}$ are strong ideals in $\mathfrak{A}$. Thus, the family $\mathfrak{J}_s(A)$ is not empty because $J_0, J_3, J_6, A \in \mathfrak{J}_s(A)$.

(b) Let $\mathfrak{C} =: (A, *_3, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. The ideal $J_1 = \{0, a\}$ is not a strong ideal in $\mathfrak{C}$ because, for example, we have $(1 *_3 a) *_3 b = b *_3 b = 0 \in J_1$ and $a \in J_1$ but $1 *_3 b = c \notin J_1$. Also, the ideal $J_3 = \{0, c\}$ is not a strong ideal in $\mathfrak{C}$ because, for example, we have $(1 *_3 c) *_3 a = a *_3 a = 0 \in J_3$ and $c \in J_3$ but $1 *_3 a = b \notin J_3$. □

Relying on the standard procedure, the following result can be proved.

Theorem 6.3. *The family $\mathfrak{J}_s(A)$ forms a complete lattice.*

Theorem 6.4. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of bounded BRK-algebras. If C is a strong ideal of $\mathfrak{B}$, then $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$.*

Proof. Since $f(0) = 0$, we have $0 \in f^{-1}(C)$.

Let $x, y, z \in A$ be such that $(x * y) * z \in f^{-1}(C)$ and $y \in f^{-1}(C)$. Then $(f(x) * f(y)) * f(z) = f((x * y) * z) \in C$ and $f(y) \in C$. Since C is a strong ideal in $\mathfrak{B}$, it follows from (Js) that $f(x * z) = f(x) * f(z) \in C$. So, $x * z \in f^{-1}(C)$. Hence $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$. □

Corollary 6.1. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism of bounded BRK-algebras. Then* $\mathrm{Ker}\, f := \{x \in A : f(x) = 0\}$ *is a strong ideal of* $\mathfrak{A}$.

Proof. Since the subset $\{0\}$ is a strong ideal in the bounded BRK-algebra $\mathfrak{B}$, by Proposition 6.2, we see that the kernel $\mathrm{Ker}\, f = f^{-1}(\{0\})$ of the homomorphism f is a strong ideal in $\mathfrak{A}$ in accordance with the previous theorem. □

We conclude this subsection with an important property of a strong ideal in a bounded BRK-algebra.

Proposition 6.3. *Let J be a strong ideal in a bounded BRK-algebra $\mathfrak{A}$. Then*

(13) $(\forall y, z \in A)((y^- * z \in J \wedge y \in F) \implies z^- \in J)$.

Proof. If we put $x = 1$ in (Js), we get (13). □

6.3. Weak ideal. The way of determining the concept of a strong ideal in a bounded BRK-algebra, as done in Definition 6.3, suggests that there is a justification for introducing the concept of a weak ideal in a BRK-algebra.

Definition 6.4. Let $A =: (\mathfrak{A}, *, 0, 1)$ be a bounded BRK-algebra. A non-empty subset J of A is a weak ideal in $\mathfrak{A}$ if it also satisfies the following condition

(Jw) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \implies x * z \in J)$.

We denote the family of all weak ideals in a bounded BRK-algebra $\mathfrak{A}$ by $\mathfrak{J}_w(A)$.

Proposition 6.4. *Any weak ideal J in a bounded BRK-algebra $\mathfrak{A}$=:(A,$*$,0,1) satisfies the condition* (J0).

Proof. Since J is an non-empty subset in A, there exists an element $x \in A$ such that $x \in J$. Then, from $x = x * (x * x) \in J$ and $x \in J$ it follows that $x * x = 0 \in J$ by (Jw). □

Proposition 6.5. *Every weak ideal in a bounded BRK-algebra $\mathfrak{A}$=: (A,$*$,0,1) is an incomplete sub-algebra in $\mathfrak{A}$.*

Proof. Let J be a weak ideal in a bounded BRK-algebra $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then, from $x * (y * y) = x * 0 = x \in J$ and

$y \in J$ it follows that $x * y \in J$ by (Jw). So, J is an incomplete sub-algebra in $\mathfrak{A}$. □

Proposition 6.6. *Every weak ideal in a bounded BRK-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 0$ in (Jw), we get (J1). □

Proposition 6.7. *Let J be a weak ideal in a bounded BRK-algebra $\mathfrak{A}$. Then*

(14) $(\forall x, z \in A)(x \in J \implies x * z \in J)$,

(15) $(\forall y, z \in A)(((y * z)^- \in J \wedge y \in J) \implies z^- \in J)$.

Proof. (14): If we put $y = 0$ in (Jw), we see that $x * (0 * z) = x * 0 = x \in J$ and $0 \in J$ imply $x * z \in J$ with respect to (L) and (M).

(15): If we put $x = 1$ in (Jw), we get (14). □

Example 6.2. (a) Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. Subsets $J_0 = \{0\}$, $J_1 =: \{0, a\}$ and $J_3 = \{0, c\}$ are weak ideals in $\mathfrak{A}$.

(b) Let $\mathfrak{C} =: (A, *_3, 0, 1)$ be a bounded BRK-algebra as in Example 3.1. The ideal $J_3 =: \{0, c\}$ is a weak ideal in $\mathfrak{C}$.

The family $\mathfrak{J}_w(A)$ is not empty because:

Proposition 6.8. *In every bounded BRK-algebra $\mathfrak{A}$, the incomplete subalgebra $\{0\}$ is a weak ideal in $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $x * (y * z) = 0$ and $y = 0$. Then $0 = x * (0 * z) = x * 0 = x$ with respect to (L) and (M). Thus $0 = x = 0 * z = x * z$ according to (L). Therefore, $\{0\}$ is a weak ideal in $\mathfrak{A}$. □

Relying on the standard procedure, the following result can be proved.

Theorem 6.5. *The family $\mathfrak{J}_w(A)$ forms a complete lattice.*

Conclusion 5. *The families $\mathfrak{J}_s(A)$ and $\mathfrak{J}_w(A)$ differ from each other because, for example, we have $J_1 \in \mathfrak{J}_w(A)$ but $J_1 \notin \mathfrak{J}_s(A)$ as shown in Examples 6.2(a) and 6.1(a).*

7. Involutive BRK-algebras

The idea of involution on logical algebras was introduced in 1979 by K. Iseki ([11]) when he raised the question of the existence of a non-commutative BCK-algebra that satisfies the so-called Double Negation Condition. An algebra that satisfies this condition is called an involutive algebra. Many involutive algebras and their internal architecture have been the subject of interest to several researchers. For example, involutive BE-algebras ([5]), involutive GE-algebras ([4]) and involutive WE-algebras ([18]) have been considered.

Definition 7.1. An algebra $\mathfrak{A} =: (A, *, 0, 1)$ is called an involutive BRK-algebra if it is a bounded BRK-algebra with the unit 1 satisfying the Double Negation condition:

(DN) $(\forall x \in A)(((x^-)^- = x)$.

Denote by $\mathbf{BRK}_{(DN)}$ the classes of involutive BRK-algebras.

Not every bounded BRK-algebra has to be an involutive BRK-algebra. For example, the bounded BRK-algebras $\mathfrak{A} =: (A,*,0,1)$ and $\mathfrak{C} =: (A,*_3,0,1)$ in Example 3.1 are not involutive BRK-algebras.

Example 7.1. The bounded BRK-algebra $\mathfrak{B} =: (A, *_2, 0, 1)$ in Example 3.1 is an involutive BRK-algebra. □

Conclusion 6. *The class $\boldsymbol{BRK}_{(ND)}$ of involutive BRK-algebras is a subclass of involutive RML-algebras. The converse is not necessarily true. For example, the RML-algebra in Example 3.2 is an involutive RML-algebra but it is not an involutive BRK-algebra.*

8. Final comments

The paper discusses the concepts of bounded and involutive BRK-algebras. It is shown that the class of bounded (involitive) BRK-algebras is a proper subclass of bounded (involutive) RML-algebras. Also, some of their important substructures such as (incomplete) sub-algebras, ideals and filters are analyzed. Author is convinced that the material presented in this article not only enriches our knowledge of the class of BRK-algebras but also opens up space for further and more extensive research of this class of logical algebras.

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