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The general theorem of Nečas admits a simple proof

Abstract. In this note, we shall study a new, short proof of the general theorem of Nečas about the solvability of linear partial differential equations in the Banach space setting. The complexity of this proof does not seem to be greater than that of the Lax–Milgram theorem, and since the theorem of Nečas is strictly stronger than the Lax–Milgram theorem, the author hopes that his new proof will help the theorem of Nečas to gain prominence in PDE and functional analysis lectures.

Theorem 1 (Nečas). *Let X, Y be Banach spaces, and let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form, where Y is reflexive*[1]*. The problem*

$$\forall v \in Y : a(u, v) = l(v)$$

has a unique solution $u \in X$ for all $l \in Y'$ if and only if the following two conditions are satisfied:

$$\exists c > 0 : \forall u \in X : c\|u\|_X \leq \sup_{v \in Y \setminus \{0\}} \frac{|a(u, v)|}{\|v\|_Y} \tag{1}$$

$$\forall v \in Y : (\forall u \in X : a(u, v) = 0) \Rightarrow v = 0 \tag{2}$$

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[1]The necessity of the two conditions is also given if Y is not reflexive, as the proof will show.

Note that if $Y = X$ and X is an inner product space wrt. a, then the first condition reduces to coercivity, and the second condition is automatically satisfied.

Here follows a new, short proof:

Proof. The unique solvability (for each l) of the given problem is equivalent to the linear operator

$$A : X \to Y', A(u) := (v \mapsto a(u, v))$$

being bijective. A is bounded because a is and by the definition of the norm of Y' (see also (3)). Let us first prove that the conditions are necessary. The necessity of the condition (1) follows immediately from the corollary to the open mapping theorem and

$$\|Au\|_{Y'} = \sup_{v \in Y \setminus \{0\}} \frac{|a(u, v)|}{\|v\|_Y}. \tag{3}$$

The necessity of the condition (2) follows from the Hahn–Banach theorem, because if A is surjective, then any v satisfying the premise vanishes on every linear functional on Y and hence must be zero.

Now for sufficiency note that the condition (1) immediately implies injectivity. In order to prove the surjectivity of A, we proceed in two steps: First, the condition (1) also implies that the range of A is closed (for if $w \in Y'$ and $A(u_n) \to w$, then $A(u_m - u_n)$ for $m > n$ goes to zero as n tends to infinity and hence the first condition implies that $(u_n)_{n \in \mathbb{N}}$ is Cauchy, hence convergent, and by continuity the limit is mapped to w), and therefore, if A is not surjective, any nonzero functional on $Y'/A(X)$, which exists due to the Hahn–Banach theorem, furnishes (after post-composition with $Y' \to Y'/A(X)$) a counter-example to the condition (2) by the reflexivity of Y. □

The sufficiency proof is very similar to the original argument of Nečas [3, p. 318f.], though it was found independently by the author.

Ern–Guermond claim the theorem to be a reformulation of the open mapping and closed graph theorems (cf. Ern–Guermond [2, p. 17]), but the above proof does not use the closed graph theorem at all. If one starts with their formulation [2, (25.14), p. 19] instead of with the operator A they give, the above proof also carries over to that situation; all we need to know in addition is that the canonical map $J : Y \to Y''$ is norm-preserving and thus an embedding onto a closed subspace of Y''.

The classical application of the theorem of Nečas are weak solutions to Poisson's equation $\Delta u = f$, but since for that equation the Lax–Milgram theorem is sufficient, the author would like to point to the stationary advection-reaction equation $f = \nu u + \nabla \cdot (\beta u)$, to be solved for scalar u, with scalar data ν and f and vector data β (cf. eg. [1, Chapter 2]).

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